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On Invariants of Complex Filiform Leibniz Algebras

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ABSTRACT

The paper intends to survey the subject of the title for an audience of mathematicians not necessarily expert in the areas of commutative algebra and algebraic geometry. It is devoted to the isomorphism invariants of low dimensional Complex filiform Leibniz Algebras under the action of the general linear group ("transport of structure"). The description of the field of invariant rational functions in low dimensional cases is presented.

Keywords: field of invariant function, group action, Leibniz algebra, orbit.
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INTRODUCTION

1. Group Action. Orbits.

Let G be a group and X be a nonempty set.

Definition 1.1. An action of the group G on X is a function $\sigma : G \times \mathrm{X} \to \mathrm{X}$ with:

(i) $\sigma(e,x) = x$, where e is the unit element of G and $x \in \mathrm{X}$

(ii) $\sigma(g, \sigma(h,x)) = \sigma(gh,x)$, for any $g,h \in G$ and $x \in \mathrm{X}$.

We shortly write gx for $\sigma(g,x)$, and call X a G-set.

Let K be a field $K[\mathrm{X}] = \{f : \mathrm{X} \to K\}$ be the set of all functions on X. It is an algebra over K with respect to point wise addition, multiplication and multiplication by scalar operations.

Definition 1.2. A function $f : \mathrm{X} \to K$ is said to be invariant if $f(gx) = f(x)$ for any $g \in G$ and $x \in \mathrm{X}$. The set of invariant functions on X, denoted by $K[\mathrm{X}]^G$, is a subalgebra of $K[\mathrm{X}]$.

$O(x) = \{ y \in X \mid \exists g \in G$ such that $y = gx \}$ is called the orbit of the element x under the action of G.

Theorem 1.3. Let X be a G-set. Define a relation ~ on X by for all $x, y \in \mathrm{X}$, $x \sim y$ if and only if $gx = y$ for some $g \in G$.

Then ~ is an equivalent relation on X.

It is evident that the equivalence classes with respect to equivalent relation ~ are the same with the orbits of the action of the group G.

If we consider the factor set X/~ then the invariant functions are functions on X/~ .

2. Affine Algebraic Variety [Hartshorne, 1977].

- Definition and examples
- Algebra of regular function on A.A.V.
- Irreducibility
- Field of rational function
- Morphism of A.A.V.

Let K be a fixed algebraically closed field. We define affine space over K denoted $\mathbf{A}^n$, to be the set of all n-tuples of elements of K. An element $\mathrm{P} \in \mathbf{A}^n$ will be called a point, and if $\mathrm{P} = (a_1,...,a_n)$ with $a_i \in K$, then the a_i will be called the coordinates of P.

Let $A = K[x_1, ..., x_n]$ be the polynomial ring in n variables over K. We will interpret the elements of A as functions from the affine n-space to K, by defining $f(\mathrm{P}) = f(a_1,..,a_n)$, where $f \in A$ and $\mathrm{P} \in \mathbf{A}^n$. Thus if $f \in A$ is a polynomial, we can talk about the set of zeros of f, namely $\mathrm{Z}(f) = \{ \mathrm{P} \in \mathbf{A}^n \mid f(\mathrm{P}) = 0 \}$. More generally, if T is any subset of A, we define the zero set of T to be the common zeros of all the elements of T:

$$Z(\mathrm{T}) = \{ \mathbf{P} \in \mathbf{A}^n \mid f(\mathbf{P}) = 0 \text{ for all } f \in \mathrm{T} \}.$$

Clearly if $\wp$ is the ideal of A generated by T, then $\mathrm{Z(T)} = \mathrm{Z}(\wp)$. Furthermore, since A is a noetherian ring, any ideal $\wp$ has a finite set of generators $f_1,..., f_r$. Thus Z(T) can be expressed as the common zeros of the finite set of polynomials $f_1,...,f_r$.

Definition 2.1. A subset Y of $\mathbf{A}^n$ is an algebraic set if there exists a subset $\mathrm{T} \subset A$ such that $\mathrm{Y} = \mathrm{Z(T)}$.

Proposition 2.2. The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.

Definition 2.3. We define the Zariski topology on A^n by taking the open subsets to be the complements of the algebraic sets. This is a topology, because according to the proposition, the intersection of two open sets is open, and the union of any family of open sets is open. Furthermore, the empty set and the whole space are both open.

Definition 2.4. A nonempty subset Y of a topological space X is irreducible if it cannot be expressed as the union $\mathrm{Y} = \mathrm{Y}_1 \cup \mathrm{Y}_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.

Example 2.5. $\mathbf{A}^1$ is irreducible, because its only proper closed subsets are finite, yet it is infinite (because K is algebraically closed, hence infinite).

Example 2.6. Any nonempty open subset of an irreducible space is irreducible and dense.

Example 2.7. If Y is an irreducible subset of X, then its closure in X is also irreducible.

Definition 2.8. An affine algebraic variety or simply affine variety is an irreducible closed subset of $\mathbf{A}^n$ (with the induced topology).

Now we need to explore the relationship between subsets of $\mathbf{A}^n$ and ideals in A more deeply. So for any subset $\mathrm{Y} \subset \mathbf{A}^n$, let us define the ideal of Y in A by $\mathrm{I(Y)} = \{ f \in A \mid f(\mathrm{P}) = 0 \text{ for all } \mathrm{P} \in \mathrm{Y} \}$.

Now we have a function Z that maps subsets A to algebraic sets, and a function I which maps subsets of $\mathbf{A}^n$ to ideals. Their properties are summarized in the following proposition.

Proposition 2.9.

(a) If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$.

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{A}^n$, then $I(Y_1) \supseteq I(Y_2)$.

(c) For any two subsets Y_1, Y_2 of $\mathbf{A}^n$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d) For any ideal $\wp \subseteq A$, $I(Z(\wp)) = \sqrt{\wp}$ the radical of $\wp$.

(e) For any subset $Y \subseteq \mathbf{A}^n$, $Z(I(Y))$ = the closure of Y.

Theorem 2.10 (Hilbert's Nullstellensatz). Let K be an algebraically closed field, let $\wp$ be an ideal in $A = K[x_1, ..., x_n]$, and let $f \in A$ be a polynomial which vanishes at all points of $Z(\wp)$. Then $f^r \in \wp$ for some integer $r > 0$.

Proof. Atiyah-Macdonald [Atiyah, Macdonald, 1969] or Zariski-Samuel [Zariski, Samuel, (1958, 1960)].

Corollary 2.11. There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbf{A}^n$ and radical ideals (i.e., ideals which are equal to their own radical) in A, given by $Y \to I(Y)$ and $\wp \to Z(\wp)$. Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.

Example 2.12. Let f be an irreducible polynomial in $A = K[x,y]$. Then f generates a prime ideal in A, since A is a unique factorization domain, so the zero set $Y = Z(f)$ is irreducible. It is called the affine curve defined by the equation $f(x,y) = 0$. If f has degree d, than Y is said to be a curve of degree d.

Example 2.13. More generally, if f is an irreducible polynomial in $A = K[x_1, \ldots, x_n]$, we obtain an affine variety $Y = Z(f)$, which is called a surface if $n = 3$, or a hypersurface if $n > 3$.

Example 2.14. A maximal ideal $\Im$ of $A=K[x_1, ..., x_n]$ corresponds to a minimal irreducible closed subset of $\mathbf{A}^n$, which must be a point, say $P = (a_1, \ldots, a_n)$. This shows that every maximal ideal of A is of the form $\Im = (x_1 - a_1, \ldots, x_n - a_n)$, for some $a_1, \ldots, a_n \in K$.

Example 2.15. If K is not algebraically closed, these results do not hold. For example, if K = R, the curve $x^2 + y^2 + 1 = 0$ in A^2 has no points.

Definition 2.16. If $Y \subseteq \mathbf{A}^n$ is an affine algebraic set, we define the affine coordinate ring $K[Y]$ (sometime $A(Y)$) of Y to be $A/I(Y)$.

Remark 2.17. If Y is an affine variety, then $A(Y)$ is an integral domain. Furthermore, $A(Y)$ is a finitely generated K-algebra. Conversely, any finitely generated K-algebra B which is a domain is the affine coordinate ring of some affine variety. Indeed, write B as the quotient of a polynomial ring $A=K[x_1,..., x_n]$ by an ideal $\wp$, and let Y = Z($\wp$).

Let X and Y be affine varieties and φ: X→Y be a function from X into Y.

Definition 2.18. The function φ:X→Y is called morphism (or regular mapping) from X into Y if $f \circ \varphi$ is a regular function on Y.

Example 2.19. Let $F_1, F_2,\ldots,F_m$ be polynomials in $K[T_1, ..., T_n]$ then $\varphi: A^n \to A^m$ defined by $\varphi(x) = \varphi(x_1,\ldots,x_n) = (F_1(x), F_2(x),\ldots,F_m(x))$ a morphism.

Example 2.20, Let X=Z(y^2-x^3+1) be affine algebraic variety in $\mathbf{A}^2$ with the coordinate system (x,y) and Y=Z((t^3-s^2+1),(r-s^2)) be affine algebraic variety in $\mathbf{A}^3$ with the coordinate system (s,t,r). Then the function φ defined by $\varphi(x,y)=(x, y, x^2)$ is a morphism from X into Y.

3. Algebraic Group Action

- Definition and Examples
- Orbits and their closures
- Induced action of an algebraic group on the algebra of regular functions
- Invariant regular functions. Algebra of regular invariant functions
- The fourteenth Hilbert problem
- The field of invariant rational function

Definition 3.1. An regular action of an algebraic group G on an affine variety X is a morphism ρ:G×X→X with

(i) $\rho(e,x)=x$, where e is the unit element of G and $x\in$X.

(ii) $\rho(g, \rho(h,x)) = \rho(gh,x)$, for any $g,h\in G$ and $x\in$X.

We shortly write gz for $\rho(g,x)$, and call X a G-variety.
Rather than presenting results we have preferred to work out some of the examples, partly well known and elementary, in order to introduce the subject and to explain the main ideas.

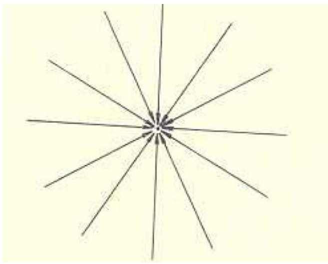
Examples. Let consider the general action of GL_2 on $\mathbf{A}^2$ defined by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{pmatrix}$$

Example 1. Let

$$G = \left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} : t \in K^* \right\}.$$

Then the orbits under this action are drawn on the picture below:



All the orbits, except for zero, are one dimensional and the closure of all contains the zero.

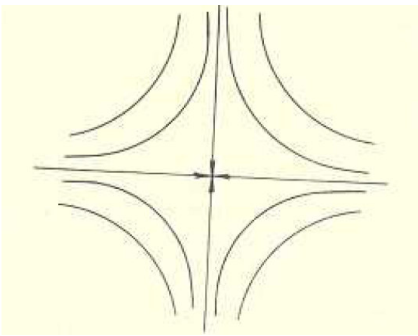
The invariant regular functions are constants only: $K[\mathrm{X}]^G = K$.
The field of invariant rational function is generated by $\frac{x}{y}$: $K(X)^G = K\left(\frac{x}{y}\right)$.

Example 2.

.

$$G = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in K^* \right\}.$$

Orbits:



All generic orbits are one dimensional and closed. There are two one dimensional orbits the closure of them contains the zero.

Invariant Regular functions: $K[\mathrm{X}]^G = K[xy]$.
Invariant Rational functions: $K(\mathrm{X})^G = K(xy)$.

Example 3. Let

$$G = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in K \right\}$$

Orbits:

Invariant Regular functions: $K[X]^G = K[y]$.
Invariant Rational function: $K(X)^G = K(y)$.

Let a group G acts on an algebraic variety X regularly. For simplicity, we assume that the base field K is algebraically closed and of characteristic zero. We can define an action of G on the K-algebra $K[X]$ of regular functions on X: $(gf)(x)=f(gx)$. Of special interest is the subalgebra of invariant functions which will be denoted by $K[X]^G$. It carries a lot of information about the orbit structure and its geometry.

The algebra of invariants was a major object of research in the last two centuries. There are a number of natural questions in this context:

- Is the invariant algebra $K[X]^G$ finitely generated as a K-algebra?
- If so, can one determine an explicit upper bound for the degrees of a system of generators of $K[X]^G$?
- Are there algorithms to calculate a system of generators and what is their complexity?
- If X is an irreducible how to describe $K(X)^G$?
- When $K(X)^G = QK[X]^G$?
- Describe the orbits closures of the action of G?

The first question is essentially HILBERT'S 14th problem, although his formulation was more general. The answer is positive for reductive groups by results of HILBERT, WEYL, MUMFORD, NAGATA and others, but negative in general due to the famous counter example of NAGATA (in 1959).

The fourteenth Hilbert problem: The problem of description of all linear algebraic groups for which the algebra of invariant regular functions is finite generated as a K-algebra.

Actually, we deal with the fourth and sixth questions of the above list for variety of non-Lie complex filiform Leibniz algebras in low dimensional case. In this case the field of rational invariant functions is described.

4. Complex filiform Leibniz algebras variety

- Leibniz algebras. Nilpotent and Filiform Leibniz Algebras
- Variety of algebras and Subvarieties
- The isomorphism action GL_n(C) on the variety of algebras (transport of structure)
- Main results

Let V be a vector space of dimension n over an algebraically closed field K (char K =0). The bilinear maps $V \times V \to V$ form a vector space $Hom(V \otimes V, V)$ of dimension n^3, which can be considered together with its natural structure of an affine algebraic variety over K and denoted by $Alg_n(K) \cong K^{n^3}$. An n-dimensional algebra L over K may be considered as an element $\lambda(L)$ of $Alg_n(K)$ via the bilinear mapping $\lambda : L \otimes L \to L$ defining an binary algebraic operation on L: let $\{e_1, e_2, \ldots, e_n\}$ be a basis of the algebra L. Then the table of multiplication of L is represented by point (γ_{ij}^k) of this affine space as follow: $\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma_{ij}^k e_k$, γ_{ij}^k are called *structural constants* of L. The linear reductive group $GL_n(K)$ acts on $Alg_n(K)$ by $(g * \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))$ ("transport of structure"). Two algebras λ_1 and λ_2 are isomorphic if and only if they belong to the same orbit under this action. It is clear that elements of the given orbit are isomorphic to each other algebras. The classification means to specify the representatives of the orbits.

Definition 4.1. An algebra L over a field K is called a *Leibniz algebra* if it satisfies the following Leibniz identity:

$$[x,[y,z]]=[[x,y],z]-[[x,z],y],$$

where $[\cdot,\cdot]$ denotes the multiplication in L. Let $Leib_n(K)$ be a subvariety of $Alg_n(K)$ consisting of all n-dimensional Leibniz algebras over K. It is invariant under the above mentioned action of $GL_n(K)$. As a subset of $Alg_n(K)$ the set $Leib_n(K)$ is specified by system of equations with respect to structural constants γ_{ij}^k:

$$\sum_{l=1}^{n}(\gamma_{jk}^{l}\gamma_{il}^{m}-\gamma_{ij}^{l}\gamma_{lk}^{m}+\gamma_{ik}^{l}\gamma_{lj}^{m})=0$$

It is easy to see that if the bracket in Leibniz algebra happens to be anticommutative then it is Lie algebra. So Leibniz algebras are "noncommutative" generalization of Lie algebras.

Further all algebras assumed to be over the field of complex numbers.
Let L be a Leibniz algebra. We put:

$$L^1=L,\quad L^{k+1}=[L^k,L],\ k\in N.$$

Definition 4.2. A Leibniz algebra L is said to be *nilpotent* if there exists an integer $s\in N,$ such that $L^1\supset L^2\supset\ldots\supset L^s=\{0\}.$ The smallest integer s for that $L^s=0$ is called *the nilindex* of L.

Definition 4.3. An n-dimensional Leibniz algebra L is said to be *filiform* if $dimL^i=n-i,$ where $2\le i\le n.$

There are two sources to get classification of non-Lie complex filiform Leibniz algebras. The first of them is the naturally graded non-Lie filiform Leibniz algebras and the another one is the naturally graded filiform Lie algebras [Gomez, Omirov, 2006]. Here we consider Leibniz algebras appearing from the naturally graded non-Lie filiform Leibniz algebras. According to the theorem presented in [Ayupov, Omirov, 2001] this class can be divided into two disjoint subclasses.

Theorem 4.4. Any $(n+1)$-dimensional complex non-Lie filiform Leibniz algebra obtained from the naturally graded filiform Leibniz algebras can be included in one of the following two classes of Leibniz algebras:

a) (The first class):

$$\begin{cases} [e_0,e_0]=e_2, & \\ [e_i,e_0]=e_{i+1}, & 1\le i\le n-1 \\ [e_0,e_1]=\alpha_3e_3+\alpha_4e_4+...+\alpha_{n-1}e_{n-1}+\theta e_n, & \\ [e_j,e_1]=\alpha_3e_{j+2}+\alpha_4e_{j+3}+...+\alpha_{n+1-j}e_n, & 1\le j\le n-2 \end{cases}$$

(omitted products are supposed to be zero)

b) (The second class):

$$\begin{cases} [e_0,e_0]=e_2, & \\ [e_i,e_0]=e_{i+1}, & 2\le i\le n-1 \\ [e_0,e_1]=\beta_3e_3+\beta_4e_4+...+\beta_ne_n, & \\ [e_1,e_1]=\gamma e_n, & \\ [e_j,e_1]=\beta_3e_{j+2}+\beta_4e_{j+3}+...+\beta_{n+1-j}e_n, & 2\le j\le n-2 \end{cases}$$

omitted products are supposed to be zero, where e_0, $e_1,\dots$, e_n is a basis.

In other words, the above theorem means that the above mentioned type of $(n+1)$-dimensional non-Lie complex filiform Leibniz algebras can be represented as a disjoint union of two subsets and the algebras from the difference classes never are isomorphic to each other. The basis in the theorem is called "adapted".

Let denote by $FLeib_{n+1}$ the variety of all non-Lie complex filiform Leibniz algebras from the first class and by $SLeib_{n+1}$ the variety of all non-Lie complex filiform Leibniz algebras from the second class. Each of them is invariant under isomorphism action ("transport of structure") of GL_{n+1}. It is known [Gomez, Omirov, 2006] that the "transport of structure" action of GL_{n+1} can be reduced to the action of subgroup of GL_{n+1} called adapted transformations. The next series of theorems are devoted to the description of the field of invariant rational function. The generators are written as a function of structural constants $\alpha_3, \alpha_4, ..., \alpha_n, \theta$ and $\beta_3, \beta_4, ..., \beta_n, \gamma$ with respect to the adapted basis.

MAIN RESULTS

For the simplification and computational purpose we establish the following notations:

$$\Delta_3 = \alpha_3,\ \Delta_4 = \alpha_4 + 2\alpha_3^2,\ \Delta_5 = \alpha_5 - 5\alpha_3^3,\ \Delta_6 = \alpha_6 + 14\alpha_3^4,\ \Delta_7 = \alpha_7 - 42\alpha_3^5,$$
$$\Theta_i = \alpha_i - \theta_i, \quad i = 4,5,6,7.$$

Theorem 4.5. The transcendental degree of the field of invariant rational functions $\mathbf{C}(FLeib_5)^G$ of $FLeib_5$ under the action of the adapted subgroup G of the group GL_5 is one and it is generated by the following function:

$$F = \left(\frac{\Delta_3}{\Delta_4}\right)^2 \Theta_4.$$

Theorem 4.6. The transcendental degree of the field of invariant rational functions $\mathbf{C}(FLeib_6)^G$ of $FLeib_6$ under the action of the adapted subgroup G of the group GL_6 is two and it is generated by the following functions:

$$F_1 = \frac{\Delta_3(\Delta_5 + 5\Delta_3\Delta_4)}{\Delta_4^2}, \qquad F_2 = \left(\frac{\Delta_3}{\Delta_4}\right)^3 \Theta_5.$$

Theorem 4.7. The transcendental degree of the field of invariant rational functions $\mathbf{C}(FLeib_7)^G$ of $FLeib_7$ under the action of the adapted subgroup G of the group GL_7 is three and it is generated by the following function:

$$F_1 = \frac{\Delta_3(\Delta_5 + 5\Delta_3\Delta_4)}{\Delta_4^2},\ F_2 = (\frac{\Delta_3^2(\Delta_6 + 6\Delta_3\Delta_5 + 9\Delta_3^2\Delta_4 + 3\Delta_4^2)}{\Delta_4^3},\ \ F_3 = \left(\frac{\Delta_3}{\Delta_4}\right)^4 \Theta_6.$$

Theorem 4.8. The transcendental degree of the field of invariant rational functions $\mathbf{C}(FLeib_8)^G$ of $FLeib_8$ under the action of the adapted subgroup G of the group GL_8 is four and it is generated by the following function:

$$F_1=\frac{\Delta_3(\Delta_5+5\Delta_3\Delta_4)}{\Delta_4^2},\quad F_2=\frac{\Delta_3^2(\Delta_6+6\Delta_3\Delta_5+9\Delta_3^2\Delta_4+3\Delta_4^2)}{\Delta_4^3},$$

$$F_3=\Delta_3^3\,\frac{\Delta_7+7\Delta_3\Delta_6+28\Delta_3\Delta_4^2+14\Delta_3^2\Delta_5+7\Delta_4\Delta_5+7\Delta_3^3\Delta_4+42\alpha_3^5}{\Delta_4^4},$$

$$F_4=\left(\frac{\Delta_3}{\Delta_4}\right)^5\Theta_7.$$

Later on $\Lambda_1=4\beta_3\beta_5-5\beta_4^2,\quad \Lambda_2=4\beta_3^2\beta_6-7\beta_4^3,\quad \Lambda_3=8\beta_3^3\beta_7-21\beta_4^4.$

Theorem 4.9. The transcendental degree of the field of invariant rational functions $\mathbf{C}(SLeib_5)^G$ of $SLeib_5$ under the action of the adapted subgroup G of the group GL_5 is one and it is generated by the following function:

$$G=\frac{\gamma}{\beta_3^2}.$$

Theorem 4.10. The transcendental degree of the field of invariant rational functions $\mathbf{C}(SLeib_6)^G$ of $SLeib_6$ under the action of the adapted subgroup G of the group GL_6 is one and it is generated by the following function:

$$G=\frac{2\beta_3\beta_4\gamma+\beta_3^2\Lambda_1}{\gamma^2}$$

Theorem 4.11. The transcendental degree of the field of invariant rational functions $\mathbf{C}(SLeib_7)^G$ of $SLeib_7$ under the action of the adapted subgroup G of the group GL_7 is two and it is generated by the following functions:

$$G_1=\frac{\Lambda_1^3}{(\Lambda_2-3\beta_4\Lambda_1+2\beta_4\gamma)^2},\ G_2=\frac{\gamma\Lambda_1^2}{(\Lambda_2-3\beta_4\Lambda_1+2\beta_4\gamma)^2}.$$

Theorem 4.12. The transcendental degree of the field of invariant rational functions $\mathbf{C}(SLeib_8)^G$ of $SLeib_8$ under the action of the adapted subgroup G of the group GL_8 is three and it is generated by the following functions:

$$G_1=\frac{\Lambda_1^3}{(\Lambda_2-3\beta_4\Lambda_1)^2},\ G_2=\frac{\Lambda_1^4(\Lambda_3-7\beta_4\Lambda_2-7\beta_4^2\Lambda_1+4\beta_3\beta_4\gamma)}{(\Lambda_2-3\beta_4\Lambda_1)},$$

$$G_3=\frac{\beta_3\gamma\Lambda_1^3}{(\Lambda_2-3\beta_4\Lambda_1)^3}.$$

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